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Differential-Algebraic Systems as Differential Equations on Manifolds*

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Dedicated to Professor K. Nickel for his Sixtieth Birthday

Abstract. Based on the theory of differential equations on manifolds, existence and uniqueness results are proved for a class of mixed systems of differential and algebraic equations as they occur in various applications. Both the autonomous and nonautonomous case are considered. Moreover, a class of algebraically incomplete systems is introduced for which existence and uniqueness results only hold on certain lower-dimensional manifolds. This class includes systems for which the application of ODE-solvers is known to lead to difficulties. Finally, some solution approach based on continuation techniques is outlined.

1. Introduction. Various applications involve mixed systems of differential and algebraic equations (DAE's). For instance, Gear's basic article [4] was stimulated by problems from network analysis and continuous system simulation. A different example occurs in the mathematical modeling of electrophoretic separation processes (see, e.g., [2]), and further DAE's are found in connection with certain problems in nonlinear mechanics.

In many cases, DAE's can be solved efficiently by means of standard numerical methods for ordinary differential equations (ODE's). This approach appears to have been introduced by Gear [4], and since then it has been used by several authors (see, e.g., [7], [8], [11] where further references may be found). But DAE's also have properties which may cause such ODE-solvers to run into difficulties or failures. In [5] and [7], some interesting results are presented about the causes of such difficulties in the case of a class of linear DAE's. The techniques used in these studies are algebraic in nature and do not provide complete information about the existence and uniqueness of solutions. This is the topic of our discussion here.

Our approach here is to consider DAE's as differential equations on a manifold. This allows for the development of an existence and uniqueness theory which in turn provides new insight into the properties of such DAE's and about some of the causes of the mentioned difficulties.

More specifically, after a summary of relevant theoretical results about differential equations on manifolds in Section 2, we present in Section 3 existence and uniqueness results for DAE's of the general form

$$F(y,t)=0, \qquad A(y,t)\,\frac{dy}{dt}=G(y,t), \tag{1.1}$$

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where F and G are nonlinear mappings and A a matrix operator. Both the autonomous and the nonautonomous case are considered. Then in Section 4 some extensions of the basic results are discussed and, in particular, it is shown that certain DAE's are algebraically incomplete and possess existence and uniqueness properties only on specific lower-dimensional manifolds. Algebraic incompleteness appears to be a central cause of the mentioned difficulties when ODE solvers are applied, and indeed all the systems considered in [5], [7], for which such difficulties are known to arise, turn out to be algebraically incomplete. Finally, Section 5 gives a brief outlook on the numerical solution of DAE's of the form (1.1) by continuation techniques.

2. Background. For ease of reference we collect in this section some basic results about vector fields and dynamical systems on manifolds (see, e.g., [1], [6]). For our purposes it suffices to consider only manifolds modelled on R^n. Thus, let $\mathscr{M}$ be a finite-dimensional Hausdorff manifold of class C^r, $r \geqslant 2$. The tangent space of $\mathscr{M}$ at $x \in \mathscr{M}$ is denoted by $T_x(\mathscr{M})$ and $T(\mathscr{M})$ is the tangent bundle. Recall that $T(\mathscr{M})$ is of class C^{r-1}.

A vector field on $\mathscr{M}$ of class C^p, $1 \leqslant p < r$, is a C^p-mapping $v\colon \mathscr{M} \to T(\mathscr{M})$ such that $v(x) \in T_x(\mathscr{M})$ for each $x \in \mathscr{M}$. For any $x_0 \in \mathscr{M}$ an integral curve of the vector field through x_0 is a mapping $\eta\colon J \to \mathscr{M}$ of class C^p on some open interval J of R^1 containing 0 such that

$$\eta'(t) = v(\eta(t)) \quad \forall\, t \in J, \eta(0) = x_0. \tag{2.1}$$

Such integral curves exist locally on $\mathscr{M}$:

THEOREM 1. *Under the stated conditions about $\mathscr{M}$ and v, there exists for any $x_0 \in \mathscr{M}$ an integral curve of class C^p through x_0. Moreover, if $\eta_1\colon J_1 \to \mathscr{M}$, $\eta_2\colon J_2 \to \mathscr{M}$ are two such integral curves of the vector field with the same initial condition x_0, then $\eta_1(t) = \eta_2(t)$ for all $t \in J_1 \cap J_2$.*

More generally, a local flow of the vector field at $x_0 \in \mathscr{M}$ is a mapping

$$\eta\colon J \times U \to \mathscr{M} \tag{2.2}$$

with the three properties:

(a) η is of class C^p, $1 \leqslant p < r$, on the product $J \times U$ of an open interval $J \subset R^1$ containing 0 and an open neighborhood U of x_0,

(b) for each $x \in U$ the mapping $\eta_x\colon J \to \mathscr{M}$, $\eta_x(t) = \eta(t, x)\ \forall\, t \in J$, is an integral curve of v through x, and

(c) for each $t \in J$ the mapping $\eta_t\colon U \to \mathscr{M}$, $\eta_t(x) = \eta(t, x)\ \forall\, x \in U$, is a diffeomorphism from U onto the open set $\eta_t(U) \subset \mathscr{M}$.

It is then readily seen that

$$\eta_{s+t} = \eta_s \circ \eta_t = \eta_t \circ \eta_s \quad \text{if } s, t, s+t \in J, \tag{2.3}$$

and that η_0 is the identity map. Moreover, such local flows exist.

THEOREM 2. *Under the stated conditions about $\mathscr{M}$ and v, there exists for any $x_0 \in \mathscr{M}$ a local flow of class C^p at x_0. Moreover, any two such local flows are equal on the intersection of their domains of definition.*

Theorem 1 implies that the union of the domains of all integral curves of v through a given point $x \in \mathscr{M}$ is an open interval $J_x = (t_x^-, t_x^+)$ where $t_x^- = -\infty$ and

$t_x^+ = +\infty$ are not excluded. Let

$$\mathscr{D}(v) = \{(t, x) \in R^1 \times \mathscr{M};\, t_x^- < t < t_x^+\}; \tag{2.4}$$

then the following global result holds.

THEOREM 3. *Under the stated conditions about $\mathscr{M}$ and v, the set $\mathscr{D}(v)$ is open in $R^1 \times \mathscr{M}$ and contains $\{0\} \times \mathscr{M}$. Moreover, there exists a unique C^p-mapping η^*: $\mathscr{D}(v) \to \mathscr{M}$ such that for any $x \in \mathscr{M}$ the mapping $\eta_x^*\colon J_x \to \mathscr{M}$, $\eta_x^*(t) = \eta^*(t, x)$ is an integral curve of v through x^*.*

The mapping η^* is called an integral of v and the curve η_x^* a maximal integral curve of v through x.

THEOREM 4. *Under the stated conditions for $\mathscr{M}$ and v, let $x \in \mathscr{M}$ be a point for which $t_x^+ < \infty$. Then for any compact set $C \subset \mathscr{M}$ there exists $\varepsilon > 0$ such that $\eta^*(t, x) \notin C$ for $t > t_x^+ - \varepsilon$.*

A corresponding result holds when $t_x^- > -\infty$.

3. Some Existence Results for DAE's. In connection with our consideration of differential-algebraic systems all manifolds turn out to be the solution sets of a finite-dimensional nonlinear equation. More specifically, we use the following basic result (see, e.g., [3]).

THEOREM 5. *Let $F\colon S \subset R^n \to R^m$, $1 \leqslant m < n$, be a C^r-mapping, $r \geqslant 1$, on an open set $S \subset R^n$. Then the regularity set*

$$\mathscr{R}(F, S) = \{x \in S, \operatorname{rge} DF(x) = R^m\} \tag{3.1}$$

is open in R^n, and for $0 \in F(\mathscr{R}(F, S))$ the regular solution set

$$\mathscr{M} = \mathscr{M}(F, S) = \{x \in \mathscr{R}(F, S), F(x) = 0\} \tag{3.2}$$

is a nonempty (sub)manifold of R^n of class C^r and dimension $n - m$.

As a first application consider the autonomous DAE

$$F(y) = 0, \qquad A(y)\frac{dy}{dt} = G(y) \tag{3.3}$$

where

$$F\colon S \to R^m, \quad A\colon S \to L(R^n, R^{n-m}), \quad G\colon S \to R^{n-m} \tag{3.4}$$

are given C^r-maps, $r \geqslant 2$, on some open set $S \subset R^n$.

If $y\colon J \subset R^1 \to R^n$ denotes a continuously differentiable solution curve of (3.3) on some open interval J, then for any $t \in J$ the tangent vector $q(t) = dy(t)/dt$ must satisfy

$$N(y(t))q(t) = \begin{pmatrix} 0 \\ G(y(t)) \end{pmatrix}, \qquad N(y) = \begin{pmatrix} DF(y) \\ A(y) \end{pmatrix}. \tag{3.5}$$

Evidently the set

$$S_0 = \{y \in S;\, N(y) \text{ is nonsingular}\} \tag{3.6}$$

is an open subset of R^n. We assume that $\mathscr{M}_0 = \mathscr{M}(F, S) \cap S_0$ is nonempty. As an open subset of $\mathscr{M}$, the set $\mathscr{M}_0$ constitutes a submanifold of $\mathscr{M}$ with the same dimension as $\mathscr{M}$ whence $T_y(\mathscr{M}_0) = T_y(\mathscr{M})$ for all $y \in \mathscr{M}_0$. Therefore, the mapping

$$(3.7) \qquad v: \mathscr{M}_0 \to T(\mathscr{M}_0), \quad v(y) = N(y)^{-1}\begin{pmatrix} 0 \\ G(y) \end{pmatrix} \quad \forall y \in \mathscr{M}_0,$$

defines a vector field on $\mathscr{M}_0$. Because the mappings (3.4) are of class C^r on S, $y \to N(y)^{-1}$ and therefore also $y \to v(y)$ are of class C^{r-1} on $\mathscr{M}_0$. Evidently, the integral curves of v on $\mathscr{M}_0$ are exactly the solutions of (3.3) in S_0.

Now the results of Section 2 are directly applicable to v on $\mathscr{M}_0$. For ease of reference, we summarize this in the following form:

THEOREM 6. *Let the maps* (3.4) *be of class* C^r, $r \geqslant 2$, *on the open set* $S \subset R^n$ *and assume that the submanifold* $\mathscr{M}_0$ *is nonempty. Then, for any* $y_0 \in \mathscr{M}_0$, *there exists on* $\mathscr{M}_0$ *a unique, maximally extended* C^{r-1}*-solution of* (3.3) *through* y_0 *which has no endpoint in* $\mathscr{M}_0$. *Moreover, the dependence of the solutions of* (3.3) *upon their initial points* $y_0 \in \mathscr{M}_0$ *is of class* C^{r-1}.

The existence and uniqueness of local solutions derives from Theorem 1, their extendability to a maximal solution from Theorem 3, while the fact that no such solution can end in $\mathscr{M}_0$ follows from Theorem 4. Finally, the C^{r-1}-dependence of the solutions on the initial conditions is a direct consequence of Theorem 2.

As a simple example we consider the system

$$(3.8) \qquad \begin{aligned} F(y) &\equiv 0.8y_1 + 1.6y_3 - 0.6y_1y_3 = 0, \\ y_1' = g_1(y) &\equiv -0.8y_1 + 10y_2 - 0.6y_1y_3, \\ y_2' = g_2(y) &\equiv -10y_2 + 1.6y_3. \end{aligned}$$

It represents a limiting case for $\varepsilon \to 0$ of a system of ODE's considered in [11] and arising in connection with the Belousov-Zhabotinskii reaction. Here the determinant of the matrix $N(y)$ in (3.5) has the value $1.6 - 0.6y_1$ and, clearly, the plane in R^3 defined by $1.6 - 0.6y_1 = 0$ does not intersect the manifold $\mathscr{M}(F, R^3)$. In other words, we have $\mathscr{M}_0 = \mathscr{M}$ and the direction field v has the components

$$v_1(y) = \frac{4 - 3y_3}{8 - 3y_1}F(y), \quad v_2(y) = g_1(y), \quad v_3(y) = g_2(y) \quad \forall y \in R^3, y_1 \neq \frac{8}{3}.$$

In particular, for any starting point $x^0 \in \mathscr{M}$, Theorem 6 ensures the existence of a unique maximal C^∞-solution of (3.8) through that point.

We turn now to a nonautonomous system

$$(3.9) \qquad F(y, t) = 0, \qquad A(y, t)\frac{dy}{dt} = G(y, t),$$

where the mappings (3.4) are of class C^r, $r \geqslant 1$, on the open set $S \subset R^n \times R^1$. A well-known technique for reducing (3.9) to the form (3.3) is the addition of the differential equation

$$(3.10) \qquad t' = 1.$$

Then the $(n + 1) \times (n + 1)$ matrix $N(y, t)$ of (3.5) equals

$$(3.11) \qquad \begin{pmatrix} D_yF(y,t) & D_tF(y,t) \\ A(y,t) & 0 \\ 0 & 1 \end{pmatrix},$$

and hence the definition of S_0 is equivalent with

$$(3.12)\quad S_0 = \left\{ (y,t) \in R^n \times R^1;\ L(y,t) \equiv \begin{pmatrix} D_y F(y,t) \\ A(y,t) \end{pmatrix} \text{ is nonsingular} \right\}.$$

The vector field on $\mathcal{M}_0 = \mathcal{M}(F, S) \cap S_0$ now becomes

$$(3.13)\quad v: \mathcal{M}_0 \to T(\mathcal{M}_0),\quad v(y,t) = L(y,t)^{-1} \begin{pmatrix} -D_t F(y,t) \\ G(y,t) \end{pmatrix} \quad \forall (y,t) \in \mathcal{M}_0,$$

and Theorem 6 applies on this submanifold $\mathcal{M}_0$ of $\mathcal{M}$.

Another approach to the treatment of (3.9) is to consider R^n as the affine part of the projective n-space (see, e.g., [1]). Instead of developing this formally, we observe simply that if $y = y(s)$, $t = t(s)$, $s \in J \subset R^1$, denotes a solution of (3.9) which is parametrized in terms of some real parameter s, then the tangent vector of this solution curve must satisfy

$$(3.14)\quad K(y(s), t(s)) \begin{pmatrix} \dot{y}(s) \\ \dot{t}(s) \end{pmatrix} = 0, \qquad K(y,t) = \begin{pmatrix} D_y F(y,t) & D_t F(y,t) \\ A(y,t) & -G(y,t) \end{pmatrix},$$

where dots represent derivatives with respect to s. If $K(y,t)$ has full rank then the one-dimensional null-space $\ker K(y,t)$ specifies a point in projective n-space. In order to define a vector field on some submanifold of $\mathcal{M}(F, S)$ we need to select a point on this one-dimensional space. One such selection was accomplished with (3.10). Another approach is to choose a specific direction and normalization of the null-vector of $K(y,t)$.

For this let

$$(3.15)\qquad S_1 = \{ (y,t) \in R^n \times R^1;\ \operatorname{rank} K(y,t) = n \}$$

and assume that $\mathcal{M}_1 = \mathcal{M}(F, S) \cap S_1$ is nonempty. Once again, since S_1 is open, $\mathcal{M}_1$ is a submanifold of the same dimension as $\mathcal{M}$ and at each point of $\mathcal{M}_1$ the tangent-manifolds are the same. Evidently, the specification

$$(3.16)\qquad K(y,t)v(y,t) = 0,\quad \|v(y,t)\|_2 = 1,\quad \det \begin{pmatrix} K(y,t) \\ v(y,t)^T \end{pmatrix} > 0$$

defines a mapping $(y,t) \in S_1 \to v(y,t) \in R^{n+1}$ which is of class C^{r-1} on S_1, (see, e.g., [9]). Clearly, for $(y,t) \in \mathcal{M}_1$ we have $v(y,t) \in T_{(y,t)}(\mathcal{M}) \equiv T_{(y,t)}(\mathcal{M}_1)$ and thus $v: \mathcal{M}_1 \to T(\mathcal{M}_1)$ represents a C^{r-1}-vector field on $\mathcal{M}_1$. Moreover, the integral curves of v on $\mathcal{M}_1$ are exactly the solutions of the homogenized system

$$(3.17)\qquad F(y,t) = 0,\quad A(y,t)\dot{y} - G(y,t)\dot{t} = 0,\quad \|\dot{y}\|_2^2 + |\dot{t}|^2 = 1$$

in the set S_1. Hence, the solutions are here parametrized curves with the arclength as parameter. Other parametrizations may be considered as well (see, e.g., [9], [10]).

Thus, once again, we may apply the theorems of Section 2, and, in analogy with Theorem 6, this gives the following result:

THEOREM 7. *Let the maps* (3.4) *be of class C^r, $r \geq 2$, on the open set $S \subset R^n \times R^1$ and assume that the submanifold $\mathcal{M}_1$ is nonempty. Then for any $(y_0, t_0) \in \mathcal{M}_1$ there exists a unique, maximally extended C^{r-1}-solution of* (3.17) *on $\mathcal{M}_1$ through the point (y_0, t_0). This solution curve has no endpoint in $\mathcal{M}_1$. Moreover, the dependence of the solutions of* (3.17) *on their initial points $(y_0, t_0) \in \mathcal{M}_1$ is of class C^{r-1}.*

Clearly, S_1 contains the set S_0 of (3.12) and hence we have $\mathscr{M}_0 \subset \mathscr{M}_1$. The points in $\mathscr{M}_1$ not in $\mathscr{M}_0$ are limit points of solutions of (3.17) with respect to the parameter t.

4. Extensions. The results of the previous section may be extended and generalized in various ways. For instance, we may consider a system of the general form

$$F(y, t) = 0, \qquad G(y', y, t) = 0 \tag{4.1}$$

where—under appropriate differentiability assumptions—the derivative of G with respect to y' has full rank. Then the implicit function theorem allows a local reduction of (4.1) to the form (3.9) with the identity matrix in place of A. Hence, from Theorem 3 we may conclude the validity of a local existence result for (4.1). We shall not dwell on the straightforward formulation of such a result.

The system (3.3) was "square," that is, the total number of algebraic and differential equations equalled the number of unknown variables. Suppose that we have a system of the same form

$$F(y) = 0, \qquad A(y)\frac{dy}{dt} = G(y) \tag{4.2}$$

where, instead of (3.4), the mappings satisfy

$$F: S \to R^{m_1}, \quad A: S \to L(R^n, R^{m_2}), \quad G: S \to R^{m_2} \tag{4.3}$$

$$1 \leqslant m_1 < n, n \geqslant m_2 \geqslant 1, m_1 + m_2 \geqslant n,$$

and again are of class C^r, $r \geqslant 2$, on some open set $S \subset R^n$. As before, the tangent vector $q = q(t)$ at any point of a solution curve of (4.2) must solve the system (3.5). But now this system may be overdetermined and hence has a solution only if its right side belongs to the range of $N(y)$.

In generalization of Theorem 6 we then obtain the following result:

THEOREM 8. *Under the stated conditions* (4.3) *about the maps let*

$$S_0 = \left\{ y \in S;\ \operatorname{rank} N(y) = n,\ \begin{pmatrix} 0 \\ G(y) \end{pmatrix} \in \operatorname{rge} N(y) \right\} \tag{4.4}$$

and suppose that there exists a nonempty set $\mathscr{M}_0 \subset \mathscr{M}(F, S) \cap S_0$ which is open in $\mathscr{M}$. Then, for any $y_0 \in \mathscr{M}_0$, there exists on $\mathscr{M}_0$ a unique, maximally extended C^{r-1}-solution of (4.2) *through y_0 which has no endpoint in $\mathscr{M}_0$. Moreover, the dependence of the solutions of* (4.2) *upon their initial points $y_0 \in \mathscr{M}_0$ is of class C^{r-1}.*

Evidently, by (4.3), S_0 and $\mathscr{M} = \mathscr{M}(F, S)$ have dimensions $2n - m_1 - m_2 > 0$ and $n - m_1 > 0$, respectively, and it follows that $2n - m_1 - m_2 \geqslant n - m_1$ which is necessary for $\mathscr{M}_0$ to be open in $\mathscr{M}$. The open set $\mathscr{M}_0$ constitutes a submanifold of $\mathscr{M}$ with the same dimension and hence, for each $y \in \mathscr{M}_0$, the vector $v(y) \in T_y(\mathscr{M}) = T_y(\mathscr{M}_0)$ specified by

$$N(y)v(y) = \begin{pmatrix} 0 \\ G(y) \end{pmatrix}, \qquad y \in \mathscr{M}_0, \tag{4.5}$$

is well-defined and introduces a vector field $v: \mathscr{M}_0 \to T(\mathscr{M}_0)$ on $\mathscr{M}_0$. It follows readily that v is of class C^{r-1} on $\mathscr{M}_0$. Clearly, the integral curves of v on $\mathscr{M}_0$ are exactly the solutions of (4.2) in that set, and hence we can apply the existence theory of Section 2.

This result also extends to the nonautonomous case (3.9) when the mappings (4.3) are of class C^r, $r \geqslant 2$, on the open set $S \subset R^n \times R^1$. As before, we either reduce the problem to the autonomous case by adding the equation (3.10) or we homogenize the equations and consider the corresponding system (3.17).

In some applications the set S_0 of (4.4) turns out to be empty. For instance, this is the case for the problem

$$(4.6) \qquad \begin{aligned} &(\mathrm{a}) \quad y_1^2 + y_2^2 = 1 \\ &(\mathrm{b}) \quad y_1' = y_3 \\ &(\mathrm{c}) \quad y_2' = y_4 \\ &(\mathrm{d}) \quad y_3' = -y_1 y_5 \\ &(\mathrm{e}) \quad y_4' = -y_2 y_5 + 1 \end{aligned}$$

considered in [5]. It describes a simple pendulum where y_1, y_2 are the distances from the pivot and y_5 is the string tension. Here, the fifth column of $N(y)$ is zero, and the right side of (4.5) is in the range of $N(y)$ exactly if

$$(4.7) \qquad y_1 y_3 + y_2 y_4 = 0.$$

As we saw, this is a necessary condition for the solvability of (4.6). Its validity, for any solution of (4.6), can also be deduced directly from the equations. Thus, instead of the four-dimensional manifold in R^5 defined by (4.6a), we have to use the three-dimensional submanifold specified by (4.6a) and (4.7). Since this reduces also the tangent-manifolds, it means simply that we have to add the equation (4.7) to the system (4.6) and form the corresponding new matrix $N(y)$. Once again, the fifth column of $N(y)$ is zero, and the right side is in the range of $N(y)$ exactly if

$$(4.8) \qquad y_3^2 + y_4^2 + y_2 - y_5 = 0.$$

This relation is not as self-evident as (4.7), but it can also be deduced for all solutions of (4.6) by differentiation of (4.7) and application of (4.6b–e). As before, (4.8) has to be added to the system, and hence our final two-dimensional manifold $\mathscr{M}$ in R^5 is now defined by the three equations (4.6a), (4.7), and (4.8). Thus the expanded system has the form (4.2) with $n = 5$, $m_1 = 3$, $m_2 = 4$. A simple calculation shows that the corresponding matrix $N(y)$ has rank 5 and that on all of $\mathscr{M}$ the right side is in the range of $N(y)$. Thus on $\mathscr{M}_0 = \mathscr{M}$, Theorem 8 applies to the augmented system and therefore also to the original system (4.6). Note that for the computation this result requires the initial point to satisfy all three equations (4.6a), (4.7), (4.8) and not only (4.6a).

This example illustrates the general procedure. For a system (4.2)–(4.3) the condition

$$(4.9) \qquad \operatorname{rank} N(y) = \operatorname{rank} \begin{pmatrix} DF(y) & 0 \\ A(y) & -G(y) \end{pmatrix}$$

specifies an algebraic relation in y which is a necessary condition for solvability. If (4.9) holds with rank $N(y) = n$ on a nonempty, open subset of our manifold then Theorem 8 can be applied. Otherwise, the relation implied by (4.9) is added to the

DAE, provided it does not reduce the differentiability-class below C^2. Since each augmentation represents a restriction to a lower-dimensional manifold, the procedure will stop at the latest after finitely many steps with a zero-dimensional manifold. In that case the system is unsolvable. It may also stop earlier, if, for instance, (4.9) applies on the entire manifold of the current system but with rank $N(y) < n$. Then there are multiple solutions. We call a differential-algebraic equation algebraically incomplete if Theorem 8 only applies after some augmentation of the algebraic equations; that is, if we can establish existence and uniqueness of solutions only on a lower-dimensional submanifold of the manifold defined by the original algebraic part of the system.

As discussed in [5], (4.6) is an example of a problem with so-called (global) index 3. Moreover, as noted there, other problems described by Euler-Lagrange equations with holonomic constraints have a corresponding property. All such problems with index larger than one are algebraically incomplete.

We illustrate this only for the simplest index 3 system

$$y_1 = g(t), \quad y_1' = y_2, \quad y_2' = y_3, \tag{4.10}$$

where $g\colon R^1 \to R^1$ is assumed to be of class C^4 on R^1. In order to bring (4.10) into autonomous form, the equation

$$t' = 1 \tag{4.11}$$

is added. The condition (4.9) is equivalent with

$$y_2 = g'(t). \tag{4.12}$$

After (4.12) is added to (4.10) the condition (4.9) for the augmented system turns out to require that

$$y_3 = g''(t). \tag{4.13}$$

In other words, our final manifold

$$\mathcal{M} = \{(y,t) \in R^3 \times R^1; y_1 = g(t), y_2 = g'(t), y_3 = g''(t)\} \tag{4.14}$$

is one-dimensional and, of course, represents by itself the unique solution of (4.10).

The situation is analogous for the generic index n problems

$$y_1 = g(t), \quad y_i' = y_{i+1}, \qquad i = 1,\dots,n-1, \tag{4.15}$$

for which $n-1$ augmentations are needed and the final manifold again has dimension one. It should be mentioned also that the augmentation process, that is, the completion of an algebraically incomplete DAE, is conceptually related to the reduction-process discussed in [5].

Finally, we note that for the homogenized form (3.17) of (4.10) the set S_1 of (3.15) is defined by

$$S_1 = \{(y,t) \in R^3 \times R^1, y_2 \neq g'(t)\}. \tag{4.16}$$

Thus, Theorem 7 applies on

$$\mathcal{M}_1 = \{(y,t) \in R^3 \times R^1; y_1 = g(t), y_2 \neq g'(t)\} \tag{4.17}$$

and the unique solution through any $(y^0, t_0) \in \mathscr{M}_1$ is $(y_1^0, y_2^0, \phi(s), t_0)^T$, $s \in R^1$, with any C^1-function ϕ on R^1. This situation arises for any system (3.9) if at a given point (y^0, t_0) the matrix $L(y^0, t_0)$ is singular but $K(y^0, t_0)$ has full rank. Then (3.17) possesses a unique solution through (y^0, t_0) for which, necessarily, $\dot{t}(y^0, t_0) = 0$. Hence, except possibly in a limiting sense, there is no solution of (3.9) through that point.

5. Outlook. The theoretical results for DAE's developed in the previous sections suggest a new numerical approach for their solution. In fact, the definitions (3.7) and especially (3.16) of the vector fields on our manifolds correspond almost directly to the definition of the vector fields underlying the general continuation processes considered, for example, in [9], and [10]. Accordingly, in place of the application of ODE-solvers we are led to a study of the possible uses of appropriate modifications of these continuation processes for the solution of DAE's.

In principle, a step of such a process would involve at least the following tasks:

(1) Compute the field-vector $v(y)$ at the current point y of the solution.

(2) If the system is in the homogenized form (3.17), determine a new local parametrization of the solution near this point.

(3) Determine a steplength and use it to compute a predicted point—usually not on the manifold—which approximates a desired point further along the solution.

(4) Start a correction iteration from the predicted point to obtain a new approximate point of the solution curve.

A simple approach is to choose the predicted point on the tangent-line $y + sv(y)$, and to use as corrector iteration the chord-Newton method applied to the system resulting from (3.17) when the derivative terms are replaced by appropriate BDF-formulas. A code using this approach has been developed as a modification of the continuation code PITCON, [10]. A description of this new DAE-solver, together with numerical results, will be given elsewhere. Preliminary results with the program have been excellent, especially, for problems where other DAE solvers, such as DASSL, [8], are running into difficulties. But further studies about the new continuation approach to the solution of DAE's are still needed. In particular, effective steplength algorithms have to be investigated which combine the requirement of a continuation process, to allow the corrector to return to the manifold, with that of an ODE solver, to control the discretization error of the differential equations.

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1. V. I. ARNOLD, *Ordinary Differential Equations*, transl. from the Russian by R. A. Silverman, The MIT Press, Cambridge, Mass., 1981.

2. M. BIER, O. A. PALUSINSKI, R. A. MOSHER & D. A. SAVILLE, "Electrophoresis: Mathematical modeling and computer simulation," *Science*, v. 219, 1983, pp. 1281–1286.

3. J. P. FINK & W. C. RHEINBOLDT, "On the discretization error of parametrized nonlinear equations," *SIAM J. Numer. Anal.*, v. 20, 1983, pp. 732–746.

4. C. W. GEAR, "Simultaneous numerical solution of differential-algebraic equations," *IEEE Trans. Circuit Theory*, v. CT-18, 1971, pp. 89–95.

5. C. W. GEAR & L. R. PETZOLD, *ODE Methods for the Solution of Differential/Algebraic Systems*, Tech. Report 82-1103, Dept. of Comp. Sci., Univ. of Illinois at Urbana-Champaign, 1982.

6. S. LANG, *Introduction to Differentiable Manifolds*, Wiley, New York, 1962.

7. L. Petzold, "Differential/algebraic equations are not ODE's," *SIAM J. Sci. Statist. Comput.*, v. 3, 1982, pp. 367–384.

8. L. R. Petzold, "A description of DASSL: A differential/algebraic system solver," in *Proc. IMACS World Congress*, Montreal, Canada, Vol. 2, R. Peskin, ed., 1982.

9. W. C. Rheinboldt, "Solution fields of nonlinear equations and continuation methods," *SIAM J. Numer. Anal.*, v. 17, 1980, pp. 221–237.

10. W. C. Rheinboldt & J. V. Burkardt, "A locally-parametrized continuation process," *ACM Trans. Math. Software*, v. 9, 1983, pp. 215–235.

11. G. Söderlind, *DASP3—A Program for the Numerical Integration of Partitioned Stiff ODE's and Differential Algebraic Systems*, Tech. Report TRITA-NA-8008, Dept. of Numer. Anal. and Comp. Science, The Royal Inst. of Technology, Stockholm, Sweden, 1980.